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Invariant star products on nondegenerate triangular Lie bialgebras over formal power series rings

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Abstract
We obtain a bijection between the set of equivalent classes of invariant star products on a non-degenerate triangular finite dimensional Lie bialgebra $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, r_t)$ over the formal power series ring $\mathbb{K}_t$ and the set $\hbar H^2(\mathfrak{a}_t)[[\hbar]]$, working in the framework developed by Etingof–Kazhdan for the quantization of Lie bialgebras. Two of the corresponding triangular Hopf algebras over the ring $\mathbb{K}_t[[\hbar]]$ are isomorphic if and only if the invariant star products defining them are equivalent. Therefore, when $t = \hbar$, we obtain a set of triangular Hopf quantized universal enveloping algebras which can also be seen as *quantizations of the deformation algebra* $\left(\mathfrak{a}_\hbar, [;]_{\mathfrak{a}_\hbar}, r_\hbar\right)$. Additionally, two of them are isomorphic if and only if the above invariant star products are equivalent.

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1. Introduction

(1) Given a Lie group endowed with a left-invariant symplectic structure we consider the set of invariant star products on it. Drinfeld quantization theory of such Lie groups proves the existence of a bijective mapping between the set of equivalent classes of those invariant star products and the set of formal powers in the deformation parameter, the 'Planck constant', with coefficients in the Chevalley second space of cohomology of the Lie algebra of the Lie group and whose first terms coincide with the given symplectic form. The purpose of this work is to obtain a *similar theorem* when the quantization of the above Lie groups is done in the framework of the theory of Lie bialgebras developed by Etingof–Kazhdan. This theory is different to the Drinfeld theory. In particular, the first theory requires us to *fix* a Lie associator and the second theory is based on the *Campbell–Hausdorff formula*. We obtain a *similar theorem* to that by Drinfeld for any nondegenerate triangular Lie bialgebra over the ring of formal power series on an indeterminate and coefficients in a field that contains the rational

numbers. The case of Lie groups can be seen as a particular case because the corresponding Lie bialgebra over the real numbers is nondegenerate triangular. Each invariant star product determines a triangular Hopf algebra and we prove that two of them are isomorphic if and only if the corresponding invariant star products are equivalent. If we identify the indeterminate of the ring of power series with the 'Planck constant' what we obtain is a set of triangular Hopf quantized universal enveloping algebras and also a necessary and sufficient condition so that two of them are isomorphic.

More explicitly,

(2) let $(\mathfrak{a}, [;]_{\mathfrak{a}})$ be a *finite dimensional* Lie algebra over a field $\mathbb{K}$ of characteristic zero. Let $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$ be a Lie algebra over the ring $\mathbb{K}_t \equiv \mathbb{K}[[t]]$ of formal power series in the indeterminate t which is a *deformation algebra* [6] of $(\mathfrak{a}, [;]_{\mathfrak{a}})$, i.e., as a $\mathbb{K}_t$-module $\mathfrak{a}_t$ is $\mathfrak{a}[[t]]$ and $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$ is $(\mathfrak{a}, [;]_{\mathfrak{a}})$ modulo t. Let u be another indeterminate. Let $(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}})$, $\mathfrak{a}_{t,u} = \mathfrak{a}_t[[u]]$, be the Lie algebra obtained by extension of the ring of scalars $\mathbb{K}[[t]] \longrightarrow \mathbb{K}_t[[u]](\mathbb{K}_{t,u} \equiv \mathbb{K}_t[[u]] \equiv \mathbb{K}[[t,u]])$ from the Lie algebra $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$. Let $\hbar$ be a third indeterminate and consider the ring $\mathbb{K}_{t,u}[[\hbar]]$. Let $(\mathfrak{a}_t[[\hbar]], [;]_{\mathfrak{a}_t[[\hbar]]})$ be the Lie algebra over $\mathbb{K}_t[[\hbar]]$ defined as before in the case of the indeterminate u.

$r_1 \in \mathfrak{a} \wedge \mathfrak{a}$ will be a given nondegenerate solution of the Yang–Baxter equation (YBE), i.e. $[r_1, r_1]_{\mathfrak{a}} = 0$, on the Lie algebra $(\mathfrak{a}, [;]_{\mathfrak{a}})$. By nondegenerate we mean rang $(r_1) = \dim \mathfrak{a}$.

The symbol $r_t = \sum_{l \geqslant 1} r_l \cdot t^{(l-1)} \in \mathfrak{a}_t \wedge \mathfrak{a}_t$, $r_l \in \mathfrak{a} \wedge \mathfrak{a}$, $l \in \mathbb{N}$, will denote a nondegenerate solution of YBE on the Lie algebra $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$.

The symbol $r_{t,u} = \sum_{l \geqslant 1} r_{t,l} \cdot u^{(l-1)} \in \mathfrak{a}_t[[u]] \wedge \mathfrak{a}_t[[u]]$, $r_{t,l} \in \mathfrak{a}_t \wedge \mathfrak{a}_t$, $l \in \mathbb{N}$, and $r_{t,1} = r_t$ will denote a nondegenerate solution of YBE on the Lie algebra $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}\right)$ over the ring $\mathbb{K}_t[[u]]$.

As $r_t \in \mathfrak{a}_t \wedge \mathfrak{a}_t$ is a solution of YBE on $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$ it defines the corresponding Poisson cohomology spaces $H^k_{P,r_t}(\mathfrak{a}_t)$. As r_t is nondegenerate, let $\mu_{r_t} : \Lambda(\mathfrak{a}_t) \longrightarrow \Lambda(\mathfrak{a}_t^*)$ be the corresponding isomorphism. Let $\mu_{r_t}(r_t) = \beta_t \in \mathfrak{a}_t^* \wedge \mathfrak{a}_t^*$ be the corresponding 2-cocycle in the Chevalley cohomology of $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$ with the trivial action of $\mathfrak{a}_t$ on K_t and $H^l(\mathfrak{a}_t)$ be the corresponding cohomological modules. Let $\overline{\mu}_{r_t} : H^l_{P,r_t}(\mathfrak{a}_t) \longrightarrow H^l(\mathfrak{a}_t)$ be the induced mapping on cohomology spaces. Similar meanings have the symbols $\mu_{r_{t,u}}, \mu_{r_1}$.

(3) $r \in (\mathfrak{a} \oplus \mathfrak{a}^*) \otimes (\mathfrak{a} \oplus \mathfrak{a}^*)$ will denote the *canonical element.* Then $r = (e_i, 0) \otimes (0, e^i)$ in any pair of dual basis. The symbol d_c will denote the co-boundary of the Chevalley cohomology of any Lie algebra *with values in the adjoint representation.* The symbol $\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}} = d_c(t,u) r\right)$ will denote the *quasitriangular double Lie bialgebra* of the *nondegenerate triangular Lie bialgebra* $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t) r_{t,u}\right)$. The element r is a solution of the YBE on $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}\right)$, and defines the symmetric element $\Omega = r + \sigma(r)$ where σ is the cycle (12). Ω is $ad_{\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}}$-invariant and it satisfies the usual infinitesimal tress relations.

(4) We fix a Lie associator $\Phi = \exp P(\hbar t_{12}, \hbar t_{23})$ over $\mathbb{K}$, [7, 8].

(5) $\left(\mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}), \Delta_0^{r_{t,u}}\right)$ will denote the universal enveloping algebra of the Lie algebra $\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}}\right)$. We do not specify its product, unit or antipode.

A theorem in [7] allows us to prove the existence of a quasitriangular quasi-Hopf algebra $\left(\mathcal{U}\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\right)[[\hbar]], \Delta_0^{r_{t,u}}, \Phi_{r_{t,u}}, R_0^{r_{t,u}} = \mathrm{e}^{\frac{\hbar}{2}\Omega}\right)$ over the ring $\mathbb{K}_{t,u}[[\hbar]]$. See theorem 2.1 .

Etingof–Kazhdan theory of quantization of Lie bialgebras [8] allows us to obtain an element $J_{r_{t,u}} \in \left(\mathcal{U}\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\right) \otimes \mathcal{U}\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\right)\right)[[\hbar]]$, verifying $J_{r_{t,u}} = 1 \otimes 1 + \frac{1}{2} r \hbar$ modulo $\hbar^2$ such that when twisting [6] the above quasitriangular quasi-Hopf algebra via the element

$J^{-1}_{r_{t,u}}$ we obtain a quasitriangular Hopf algebra $\big(\mathcal{U}\big(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\big)[[\hbar]], \Delta^{r_{t,u}}, R^{r_{t,u}}\big)$ over the ring $\mathbb{K}_{t,u}[[h]]$. We also write this algebra as $A_{(\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}})[[\hbar]],\Omega, J^{-1}_{r_{t,u}}}$.

(6) From a theorem by Etingof–Kazhdan we can obtain an element $\tilde{J}_{r_{t,u}} = (\tilde{\pi}_{t,u} \otimes \tilde{\pi}_{t,u}) J_{r_{t,u}} \in (\mathcal{U}\mathfrak{a}_{t,u} \otimes \mathcal{U}\mathfrak{a}_{t,u})[[\hbar]]$ verifying the condition $\tilde{J}_{r_{t,u}} = 1 \otimes 1 + \frac{1}{2} r_{t,u}\hbar$ modulo $\hbar^2$ such that when twisting via the element $\tilde{J}^{-1}_{r_{t,u}}$ the *trivial* triangular Hopf algebra $\big(\mathcal{U}\mathfrak{a}_{t,u}[[\hbar]], \Delta_{\mathfrak{a}_{t,u}}, R_{\mathfrak{a}_{t,u}}{=}1\otimes 1\big)$ over the ring $\mathbb{K}_{t,u}[[\hbar]]$ one obtains a *triangular* Hopf algebra $\big(\mathcal{U}\mathfrak{a}_{t,u}[[\hbar]], \tilde{\Delta}_{\mathfrak{a}_{t,u}}, \tilde{R}_{\mathfrak{a}_{t,u}}\big)$ over the ring $\mathbb{K}_{t,u}[[\hbar]]$. We will denote this algebra as $A_{\mathfrak{a}_{t,u}[[\hbar]], \tilde{J}^{-1}_{r_{t,u}}}$. The element $\tilde{J}_{r_{t,u}}$ is an invariant star product on the nondegenerate triangular Lie bialgebra $\big(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t) r_{t,u}\big)$ over the ring $\mathbb{K}_{t,u}$, see [1, 5, 13].

(7) Now, *it has a meaning to put* $u = \hbar$ in every element appearing in the definition of the quasitriangular quasi-Hopf algebra, quasitriangular Hopf algebra or triangular Hopf algebra over $\mathbb{K}_{t,u}[[\hbar]]$ considered in (5) and (6). In this way, we obtain, respectively, (i) a quasitriangular quasi-Hopf algebra $\big(\mathcal{U}\big(\mathfrak{a}_{t,\hbar} \oplus \mathfrak{a}^*_{r_{t,\hbar}}\big)[[\hbar]], \Delta_0^{r_{t,\hbar}}, \Phi_{r_{t,\hbar}}, R_0^{r_{t,\hbar}} = e^{\frac{\hbar}{2}\Omega}\big)$ over the ring $\mathbb{K}_t[[\hbar]]$; (ii) a quasitriangular Hopf algebra $\big(\mathcal{U}\big(\mathfrak{a}_{t,\hbar} \oplus \mathfrak{a}^*_{r_{t,\hbar}}\big)[[\hbar]], \Delta^{r_{t,\hbar}}, R^{r_{t,\hbar}}\big)$ over the ring $\mathbb{K}_t[[\hbar]]$; it can be obtained by a twist via the element $J_{r_{t,\hbar}} \in \big(\mathcal{U}\big(\mathfrak{a}_{t,\hbar}\oplus\mathfrak{a}^*_{r_{t,\hbar}}\big)\otimes\mathcal{U}\big(\mathfrak{a}_{t,\hbar}\oplus\mathfrak{a}^*_{r_{t,\hbar}}\big)\big)[[\hbar]]$ from that obtained in (i); and (iii) a triangular Hopf algebra $\big(\mathfrak{a}_{t,\hbar}, \tilde{\Delta}_{\mathfrak{a}_{t,\hbar}}, \tilde{R}_{\mathfrak{a}_{t,\hbar}}\big)$ over the ring $\mathfrak{a}_t[[\hbar]]$. We say that this algebra is a quantization of the pair $(\mathfrak{a}_t, r_t)$. It can be obtained by a twist via the element $\tilde{J}^{-1}_{r_{t,\hbar}} \in (\mathcal{U}\mathfrak{a}_{t,\hbar} \otimes \mathcal{U}\mathfrak{a}_{t,\hbar})[[\hbar]]$ from the *trivial* triangular Hopf algebra $\big(\mathcal{U}\mathfrak{a}_{t,\hbar}[[\hbar]], \Delta_{\mathfrak{a}_{t,\hbar}}, R_{\mathfrak{a}_{t,\hbar}} = 1 \otimes 1\big)$ over the ring $\mathbb{K}_t[[\hbar]]$.

(8) The adjoint representation of a Lie group $\mathbf{G}$ with Lie algebra $(\mathfrak{a}, [;]_{\mathfrak{a}})$ and $\mathbb{K} = \mathbb{R}$ induces a representation on the Chevalley complex $H^*(\mathfrak{a})$ that is *trivial*. This classical theorem inspired us for considering the Lie algebra isomorphisms in section 5 that allow us to obtain, in sections 6 and 7, the equivalence of invariant star products. This equivalence allows us to obtain the corresponding isomorphisms for Hopf algebras.

(9) In [17], we considered the problem of classification of invariant star products (ISPs) on a nondegenerate triangular finite dimensional Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1), r_1 \in \mathfrak{a} \wedge \mathfrak{a}$ over the field of real, $\mathbb{R}$, or complex, $\mathbb{C}$, numbers. In the notations and perspective of the present paper it corresponds to considering the classification problem of ISPs on the *trivial* nondegenerate triangular finite dimensional *deformation* Lie bialgebra $(\mathfrak{a}\otimes\mathbb{K}[[t]], [,]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c r_1), r_1 \in \mathfrak{a} \wedge \mathfrak{a} \subset \mathfrak{a}_t \wedge \mathfrak{a}_t$. *Trivial* means that the Lie algebra structure of $(\mathfrak{a}_t, [,]_{\mathfrak{a}_t})$ over the ring $\mathbb{K}[[t]]$ is just the Lie algebra obtained by extension of scalars $\mathbb{K} \longrightarrow \mathbb{K}[[t]]$ from the Lie algebra structure of $(\mathfrak{a}, [,]_{\mathfrak{a}})$ over $\mathbb{K}$. In the present paper we consider the classification problem for ISPs on *any* nondegenerate triangular finite dimensional *deformation* Lie bialgebra $(\mathfrak{a}_t, [,]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c r_t), r_t = r_1 + \sum_{l>2} r_l \hbar^{(l-1)} \in \mathfrak{a}_t \wedge \mathfrak{a}_t$ of $(\mathfrak{a}, [,]_{\mathfrak{a}})$. As $\mathbb{K}[[t]]$-modules, $\mathfrak{a}_t$ is, in both cases, $\mathfrak{a}[[t]]$, but, in the general case, $(\mathfrak{a}_t, [,]_{\mathfrak{a}_t})$ is *any* Lie algebra *deformation* of $(\mathfrak{a}, [,]_{\mathfrak{a}})$. When we are dealing with the general case *we need to adapt* the quantization theory given in [8], part I, because it is based on a field $\mathbb{K}$ and not on a ring $\mathbb{K}[[t]]$. This is the reason why in [17] we had to consider the power series $r_u = r_1 + \sum_{l>2} r_l u^{(l-1)}$ appearing there as convergent series in the usual sense on the vector space $\mathfrak{a} \wedge \mathfrak{a}$ over $\mathbb{R}$ or $\mathbb{C}$ and not in the adic sense of the present paper for any $\mathbb{K}$ containing $\mathbb{Q}$. In the general case, it is also necessary to know the $\mathbb{K}[[t]]$ Hochschild cohomology modules $H^*(\mathcal{U}\mathfrak{a}_t)$ for the coalgebra structure of the universal enveloping algebra $\mathcal{U}\mathfrak{a}_t$ over $\mathbb{K}[[t]]$, which is not just the extension by scalars $\mathbb{K} \longrightarrow \mathbb{K}[[t]]$. A theorem by Cartier in [2] determines the Hochschild cohomology A-module $H^*(\Gamma M)$ for the *divided powers bialgebra* (in H Cartan sense), ΓM, of any *free* A-module, where A is any ring. The result is $H^*(\Gamma M) \simeq \Lambda(M)$, the exterior product A-module of M.

In case the ring A is a $\mathbb{Q}$-algebra, as it is the ring $\mathbb{K}[[t]]$, and M is free, the bialgebra ΓM is isomorphic to the bialgebra $TS(M)$ of *symmetric tensors* over M. In our case we take the $\mathbb{K}[[t]]$-module $\mathfrak{a}_t$. Again, as $\mathbb{K}[[t]]$ is a $\mathbb{Q}$-algebra the symmetric bialgebra $S\mathfrak{a}_t$ is isomorphic to the $\mathbb{K}[[t]]$ bialgebra $TS\mathfrak{a}_t$ of *symmetric tensors* over $\mathfrak{a}_t$. We use this last isomorphism in theorem 2.3. Then, we obtain the $\mathbb{K}[[t]]$-module isomorphism $H^*(TS\mathfrak{a}_t) \simeq \Lambda(\mathfrak{a}_t)$. Again, the coalgebra $TS\mathfrak{a}_t$ over $\mathbb{K}[[t]]$ is isomorphic to the coalgebra $\mathcal{U}\mathfrak{a}_t$ over $\mathbb{K}[[t]]$. We then have the isomorphism of $\mathbb{K}[[t]]$-modules that we need: $H^*(\mathcal{U}\mathfrak{a}_t) \simeq \Lambda(\mathfrak{a}_t)$. We may also obtain this theorem by the pattern in Drinfeld [6] and developed in [11] after Lazard 'Analyseurs' theory in [3] to compute $H^*(\mathcal{U}\mathfrak{a})$. In the proof of proposition 4.1, the theorem about the cohomological interpretation of the quantum Yang–Baxter equation in the non-trivial case $(\mathfrak{a}_t, [,]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c r_t)$ is needed. This theorem was given in [13] *for the trivial case.* It remains true in the *nontrivial* case and we may then apply it in this paper. Proposition 7.1 has to be seen as in propositions 3.7 and 3.9 in [6] but *considering at the basis* the Lie algebra $(\mathfrak{a}_t, [,]_{\mathfrak{a}_t})$ over $\mathbb{K}[[t]]$.

The proofs of these results will appear in a forthcoming paper. References [18] and [16] are related to the subject of this paper.

2. Quantization of the quasitriangular Lie bialgebra $(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}} = d_c(t,u)r)$ over the ring $\mathbb{K}_{t,u}$

(1) Pentagon, hexagon properties of associators [7, 8] and the ad-invariance of Ω allow us [7] to obtain the following:

Theorem 2.1. *Let* $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\right)$ *and* $\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}} = d_c(t,u)r\right)$ *be as in (2) of section 1. Consider the* $\mathbb{K}_{t,u}[[\hbar]]$*-module* $\mathcal{U}\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\right)[[\hbar]]$. *The set*

$$\left(\mathcal{U}\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\right)[[\hbar]], \Delta_0^{r_{t,u}}, \Phi_{r_{t,u}}, R_0^{r_{t,u}} = \mathbf{e}^{\frac{\hbar}{2}\Omega}\right)$$

is then a quasitriangular quasi-Hopf algebra over $\mathbb{K}_{t,u}[[\hbar]]$.

We do not specify the corresponding antipode. Its existence follows from theorem 1.6 in [6] and specific forms for it may be obtained from propositions 1.1 and 1.3 in [6]. We also do not specify product, unity or co-unity.

Definition 2.2. *We say that the quasitriangular quasi-Hopf algebra over* $\mathbb{K}_{t,u}[[\hbar]]$ *of theorem 2.1 is a quantization of the pair* $\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, \Omega\right)$ *or that this pair is the classical limit of the quasitriangular quasi-Hopf algebra.*

(2) Part (2) of the following theorem can be proved analogously to the corresponding one in [8] part I. We only need to remark that $\mathbb{K}_{t,u}$ is a $\mathbb{Q}$-algebra and that the symmetric algebras of the $\mathbb{K}_{t,u}$-modules $\mathfrak{a}_{t,u}, \mathfrak{a}^*_{r_{t,u}}, \left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}\right)$ are isomorphic to the corresponding algebras of *symmetric tensors* [2]. Then we apply, for example, corollary 3 of theorem 1 of section 2.8, chapter III, of [4].

Theorem 2.3. *Let* $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\right)$ *and* $\left(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}} = d_c(t,u)r\right)$ *be as in the above theorem. Let* $M(t,u)_\pm$ *be the* $\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}$*-modules with one generator* $1_\pm$ *and defined as follows:* $M(t,u)_+ = \mathcal{U}\mathfrak{a}^*_{r_{t,u}} \cdot 1_+; \mathcal{U}\mathfrak{a}_{t,u} \cdot 1_+ = 0$ *and* $M(t,u)_- = \mathcal{U}\mathfrak{a}_{t,u} \cdot 1_-; \mathcal{U}\mathfrak{a}^*_{r_{t,u}} \cdot 1_- = 0$. *Then*

(1) The equalities $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$ *define unique* $\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}$*-module morphisms* $i_\pm : M(t,u)_\pm \longrightarrow M(t,u)_\pm \otimes_{\mathbb{K}_{t,u}} M(t,u)_\pm$.

(2) *The equality* $\phi_{r_{t,u}}(1) = 1_+ \otimes 1_-$ *defines a unique* $\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}$*-module morphism* $\phi_{r_{t,u}} : \mathcal{U}(\mathfrak{a}_r \oplus \mathfrak{a}^*_{r_{t,u}}) \longrightarrow M(t,u)_+ \otimes M(t,u)_-$. *Moreover* $\phi_{r_{t,u}}$ *is an isomorphism.*

(3) *There exists an element* $J_{r_{t,u}} \in (\mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}})[[\hbar]])^{\hat{\otimes}2}$ *such that, when twisting via* $J^{-1}_{r_{t,u}}$ *the quasitriangular quasi-Hopf algebra considered in theorem 2.1, one obtains a quasitriangular Hopf algebra,* $(\mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}})[[\hbar]], \Delta^{r_{t,u}}, R^{r_{t,u}})$, *over the ring* $\mathbb{K}_{t,u}[[\hbar]]$. *The element* $J_{r_{t,u}}$ *is given by*

$$J_{r_{t,u}} = (\phi^{-1}_{r_{t,u}} \otimes \phi^{-1}_{r_{t,u}})(\Phi^{-1}_{1,2,34} \circ \Phi_{2,3,4} \circ \sigma_{23} \circ \mathrm{e}^{\frac{\hbar}{2}\Omega_{23}} \circ \Phi^{-1}_{2,3,4} \circ \Phi_{1,2,34} \circ (i_+ \otimes i_-)(\phi_{t,u}(1))),$$

and

$$\Delta^{r_{t,u}}(b) = J^{-1}_{r_{t,u}} \cdot_{r_{t,u}} \Delta_0^{r_{t,u}}(b) \cdot_{r_{t,u}} J_{r_{t,u}};\ R^{r_{t,u}} = \sigma(J^{-1}_{r_{t,u}}) \cdot_{r_{t,u}} \mathrm{e}^{\frac{\hbar}{2}\Omega} \cdot_{r_{t,u}} J_{r_{t,u}}.$$

We also have $J_{r_{t,u}} = 1 \otimes 1 + \frac{1}{2} r\hbar \mod \hbar^2$ *and* $R^{r_{t,u}} = 1 \otimes 1 + r\hbar \mod \hbar^2$. *The isomorphism* $\Phi_{r_{t,u}}$ *verifies the following equality:*

$$\Phi_{r_{t,u}} \cdot_{r_{t,u}} (\Delta_0^{r_{t,u}} \otimes id)(J_{r_{t,u}}) \cdot_{r_{t,u}} (J_{r_{t,u}} \otimes 1) = (1 \otimes \Delta_0^{r_{t,u}})(J_{r_{t,u}}) \cdot_{r_{t,u}} (1 \otimes J_{r_{t,u}}).$$

The products in these expressions are those of the enveloping algebra $\mathcal{U}((\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}) \otimes_{\mathbb{K}_{t,u}} \mathbb{K}_{t,u}[[\hbar]]) \equiv \mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}) \otimes_{\mathbb{K}_{t,u}} \mathbb{K}_{t,u}[[\hbar]]$ *defined by extension of scalars* $\mathbb{K}_{t,u} \longrightarrow \mathbb{K}_{t,u}[[\hbar]]$. *This quasitriangular Hopf algebra over* $\mathbb{K}_{t,u}[[\hbar]]$ *will be denoted by* $A_{(\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}})[[\hbar]],\Omega,J^{-1}_{r_{t,u}}}$.

Definition 2.4. *We say that the quasitriangular Hopf algebra over* $\mathbb{K}_{t,u}[[\hbar]]$ *considered in (3) of theorem 2.3 is a quantization of the pair* $(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, r)$ *or that the pair* $(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, r)$ *is the classical limit of the quasitriangular Hopf algebra over* $\mathbb{K}_{t,u}[[\hbar]]$.

Fix an ordered basis $\{e_a\}$ in $\mathfrak{a}_{t,u}$, and its dual basis $\{e^a\}$ in $\mathfrak{a}^*_{r_{t,u}}$. Then we may construct ordered bases in $\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}, \mathcal{U}\mathfrak{a}_{t,u}, \mathcal{U}\mathfrak{a}^*_{r_{t,u}}$ and $\mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}})^{\otimes 2}$.

Lemma 2.5. *The element* $J_{r_{t,u}} \in \mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}})^{\otimes^2}[[\hbar]]$ *considered in theorem 2.3, (3) has the form*

$$J_{r_{t,u}} = 1 \otimes 1 + \frac{1}{2} r\,\hbar + \sum_{k \geqslant 2} (r_{t,u}^{i_1 j_1} \ldots r_{t,u}^{i_{l(k)} j_{l(k)}} Q_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k})\hbar^k,$$

where $Q_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k} \in \mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}})^{\otimes 2}$ *are linear combinations of elements in the ordered basis fixed above. The coefficients of these linear combinations are* $\mathbb{K}$*-linear combinations of elements determined by the structure constants of the Lie algebra* $(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}})$. *The element* $r_{t,u}$ *is present in every element of the ordered basis through the product in* $\mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}})$, *but it does not occur in the coefficients defining* $Q_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k}$.

3. Quantization of the nondegenerate triangular Lie bialgebra $(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t) r_{t,u})$

As in [9, 8], we define the mapping $\chi_{r_{t,u}} : \mathfrak{a}^*_{r_{t,u}} \longrightarrow \mathfrak{a}_{t,u}$ by $\chi_{r_{t,u}}(\xi) = (\xi \otimes 1) r_{t,u}$.

Proposition 3.1. *The mapping* $\tilde{\pi}_{t,u} : \mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}} \longrightarrow \mathfrak{a}_{t,u}$, *defined by* $\tilde{\pi}_{t,u}(x;\xi) = x + \chi_{r_{t,u}}(\xi)$, *is a Lie-bialgebra morphism. That is, a Lie-algebra morphism verifying* $d_c(t) r_{t,u} \circ \tilde{\pi}_{t,u} = (\tilde{\pi}_{t,u} \otimes \tilde{\pi}_{t,u}) \circ d_c(t,u) r$. *Moreover* $(\tilde{\pi}_{t,u} \otimes \tilde{\pi}_{t,u}) r = r_{t,u}$. *The symbol* $\tilde{\pi}_{t,u}$ *will also denote the unique algebra morphism* $\tilde{\pi}_{t,u} : \mathcal{U}(\mathfrak{a}_{t,u} \oplus \mathfrak{a}^*_{r_{t,u}}) \longrightarrow \mathcal{U}\mathfrak{a}_t$ *defined by the Lie algebra morphism* $\tilde{\pi}_{t,u}$.

Theorem 3.2. *Consider the quasitriangular double Lie bialgebra* $\big(\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}} = d_c(t,u)r\big)$ *over* $\mathbb{K}_{t,u}$. *Let* $\big(\mathcal{U}\mathfrak{a}_{t,u}, \Delta_{\mathfrak{a}_{t,u}}\big)$ *be the usual Hopf universal enveloping algebra. Let*

$$\big(\mathcal{U}\big(\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}\big)[[\hbar]], \Delta_0^{r_{t,u}}, \Phi_{r_{t,u}}, R_0^{r_{t,u}} = \mathrm{e}^{\frac{\hbar}{2}\Omega}\big)$$

be the quasitriangular quasi-Hopf algebra, considered in theorem 2.1, whose classical limit is the pair $\big(\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}, \Omega\big)$. *Then we have* $(\tilde{\pi}_{t,u}\otimes\tilde{\pi}_{t,u})\circ\Delta_0^{r_{t,u}} = \Delta_{\mathfrak{a}_{t,u}}\circ\tilde{\pi}_{t,u}$. *Defining* $\tilde{\Phi}_{r_{t,u}} = (\tilde{\pi}_{t,u}\otimes\tilde{\pi}_{t,u}\otimes\tilde{\pi}_{t,u})\Phi_{r_{t,u}}$ *and* $R_{\mathfrak{a}_{t,u}} = (\tilde{\pi}_{t,u}\otimes\tilde{\pi}_{t,u})R_0^{r_{t,u}}$, *we get* $\tilde{\Phi}_{r_{t,u}} = 1\otimes 1$ *and* $R_{\mathfrak{a}_{t,u}} = 1\otimes 1$. *In this way, we obtain the (trivial) triangular Hopf algebra* $(\mathcal{U}\mathfrak{a}_{t,u}[[\hbar]], \Delta_{\mathfrak{a}_{t,u}}, \tilde{\Phi}_{r_{t,u}} = 1\otimes 1\otimes 1, R_{\mathfrak{a}_{t,u}} = 1\otimes 1)$ *over the ring* $\mathbb{K}_{t,u}[[\hbar]]$. *We call this algebra a quantization of the pair* $(\mathfrak{a}_{t,u}, 0)$, *see [6].*

From proposition 3.1, theorem 3.2 and (3) of theorem 2.3 we obtain

Corollary 3.3. *Write* $\tilde{J}_{r_{t,u}} = (\tilde{\pi}_{t,u}\otimes\tilde{\pi}_{t,u})J_{r_{t,u}} \in (\mathcal{U}\mathfrak{a}_{t,u}\otimes\mathcal{U}(\mathfrak{a}_{t,u})[[\hbar]]$. *Then*

(1) $\tilde{J}_{r_{t,u}} = 1\otimes 1 + \frac{1}{2}r_{t,u}\hbar + \cdots$

(2) $\big(\Delta_{\mathfrak{a}_{t,u}}\otimes 1\big)\tilde{J}_{r_{t,u}}\cdot\big(\tilde{J}_{r_{t,u}}\otimes 1\big) = \big(1\otimes\Delta_{\mathfrak{a}_{t,u}}\big)\tilde{J}_{r_{t,u}}\cdot\big(1\otimes\tilde{J}_{r_{t,u}}\big)$.

(3) $\tilde{R}_{\mathfrak{a}_{t,u}} = (\tilde{\pi}_{t,u}\otimes\tilde{\pi}_{t,u})R^{r_{t,u}} = \sigma\big(\tilde{J}^{-1}_{r_{t,u}}\big)\cdot(1\otimes 1)\cdot\tilde{J}_{r_{t,u}} = 1\otimes 1 + r_{t,u}\hbar + \cdots$.

The products in these expressions coincide with the products of the enveloping algebra $\mathcal{U}\mathfrak{a}_{t,u}[[\hbar]] \equiv \mathcal{U}\mathfrak{a}_t[[u,\hbar]]$. *The set* $\big(\mathcal{U}\mathfrak{a}_{t,u}[[\hbar]], \tilde{\Delta}_{\mathfrak{a}_{t,u}}, \tilde{R}_{\mathfrak{a}_{t,u}}\big)$, *denoted by* $A_{\mathfrak{a}_{t,u}[[\hbar]],\tilde{J}^{-1}_{r_{t,u}}}$, *is a triangular Hopf algebra over* $\mathbb{K}_{t,u}[[\hbar]]$. *This algebra can be obtained by a twist via the element* $\tilde{J}^{-1}_{r_{t,u}}$ *from the trivial triangular Hopf algebra* $\big(\mathcal{U}\mathfrak{a}_{t,u}\Delta_{\mathfrak{a}_{t,u}}, R_{\mathfrak{a}_{t,u}} = 1\otimes 1\big)$ *considered in theorem 3.2. It is a quantization of the pair* $(\mathfrak{a}_{t,u}; r_{t,u})$.

From lemma 2.5 we obtain

Lemma 3.4. *The element* $\tilde{J}_{r_{t,u}}$ *has the form*

$$\tilde{J}_{r_{t,u}} = 1\otimes 1 + \frac{1}{2}r_{t,u}\hbar + \sum_{k\geqslant 2}\big(r_{t,u}^{i_1 j_1}\ldots r_{t,u}^{i_{l(k)} j_{l(k)}} M_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k}\big)\hbar^k,$$

where $M_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k}$ *is a linear combination of elements in the ordered basis chosen in* $\mathcal{U}\mathfrak{a}_{t,u}$ *and whose coefficients are* $\mathbb{K}$*-linear combinations of elements (polynomials) determined by the structure constants of the Lie algebra* $\big(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}\big)$. *The element* $r_{t,u}$ *does not appear in* $M_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k} \in \big(\mathcal{U}\mathfrak{a}_{t,u}\big)^{\otimes 2}$.

We now define invariant star products.

Definition 3.5 [1, 5, 14]. *An invariant star product on a nondegenerate triangular Lie bialgebra* $\big(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\big)$ *over the ring* $\mathbb{K}_{t,u}$ *is any element* $F(t,u) = \sum_0^\infty F_k(t,u)\cdot\hbar^k \in \big(\mathcal{U}\mathfrak{a}_{t,u}\otimes\mathcal{U}\mathfrak{a}_{t,u}\big)[[\hbar]]$ *verifying the following equalities:*

(1) $F(t,u) = 1\otimes 1 \bmod \hbar$;

(2) $F(t,u) - \sigma(F(t,u)) = r_{t,u}\,\hbar \bmod \hbar^2$;

(3) $(\Delta_{\mathfrak{a}_{t,u}}\otimes 1)F(t,u)\cdot(F(t,u)\otimes 1) = (1\otimes\Delta_{\mathfrak{a}_{t,u}})F(t,u)\cdot(1\otimes F(t,u))$.

The products in (3) coincide with the products of the enveloping algebra $\mathcal{U}\mathfrak{a}_{t,u}[[\hbar]] \equiv \mathcal{U}\mathfrak{a}_t[[u,\hbar]]$, *that is, they coincide with the* $\mathbb{K}[[u,\hbar]]$ *linear extension of the product of the enveloping algebra* $\mathcal{U}\mathfrak{a}_t$.

Then we have

Proposition 3.6. *The element $\tilde{J}_{r_{t,u}} \in (\mathcal{U}\mathfrak{a}_{t,u})^{\otimes 2}[[\hbar]]$, considered in corollary 3.3, is an invariant star product on the nondegenerate triangular Lie bialgebra $\big(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\big)$ over the ring $\mathbb{K}_{t,u}$.*

Definition 3.7 [1, 5, 14]**.** *An invariant star product on a nondegenerate triangular Lie bialgebra $\big(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t\big)$ over the ring $\mathbb{K}_t$ is any element $F(t) = \sum_0^\infty F_k(t)\cdot\hbar^k \in (\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t)[[\hbar]]$ verifying the following equalities:*

(1) $F(t) = 1 \otimes 1 \;mod\, \hbar$;

(2) $F(t) - \sigma(F(t)) = r_t\hbar \;mod\, \hbar^2$;

(3) $(\Delta_{\mathfrak{a}_t} \otimes 1)F(t)\cdot(F(t)\otimes 1) = \big(1 \otimes \Delta_{\mathfrak{a}_t}\big)F(t)\cdot(1 \otimes F(t))$.

The products in (3) coincide with the products of the enveloping algebra $\mathcal{U}(\mathfrak{a}_t \otimes_{\mathbb{K}_t} \mathbb{K}[[t,\hbar]])$, that is, they coincide with the $\mathbb{K}[[\hbar]]$ linear extension of the product of the enveloping algebra $\mathcal{U}\mathfrak{a}_t$.

Proposition 3.8. *Let $F(t,u) \in \big(\mathcal{U}\mathfrak{a}_{t,u} \otimes \mathcal{U}\mathfrak{a}_{t,u}\big)[[\hbar]]$ be an invariant star product on the nondegenerate triangular Lie bialgebra $(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u})$ over the ring $\mathbb{K}_{t,u}$. Consider the element $F(t) \in (\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t)[[\hbar]]$ obtained from $F(t,u)$ by setting $u = \hbar$ in all the elements defining $F(t,u)$; in particular by setting $r_{t,\hbar} = \sum_{l\geqslant 1} r_{t,l}\hbar^l \in \mathfrak{a}_t[[\hbar]] \wedge \mathfrak{a}_t[[\hbar]]$. Then the element $F(t) \in (\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t)[[\hbar]]$ is an invariant star product on the triangular nondegenerate Lie bialgebra $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ over the ring $\mathbb{K}_t$.*

Obviously we have

Corollary 3.9. *Let $\tilde{J}_{r_{t,\hbar}} \in (\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t)[[\hbar]]$ be the element as in proposition 3.8 obtained from the element $\tilde{J}_{r_{t,u}} \in \big(\mathcal{U}\mathfrak{a}_{t,u} \otimes \mathcal{U}\mathfrak{a}_{t,u}\big)[[\hbar]]$ considered in corollary 3.3. Then $\tilde{J}_{r_{t,\hbar}}$ is an invariant star product on the triangular nondegenerate Lie bialgebra $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ over the ring $\mathbb{K}_t$.*

4. An invariant star product $F(t) \in (\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t)[[\hbar]]$ on $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ determines an element $r_{t,\hbar} \in (\mathfrak{a}_t \wedge \mathfrak{a}_t)[[\hbar]]$ such that $F(t)$ and $\tilde{J}_{r_{t,\hbar}} \in (\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t)[[\hbar]]$ are equivalent

Let $F(t) \in \mathcal{U}\mathfrak{a}_t[[\hbar]] \hat{\otimes} \mathcal{U}\mathfrak{a}_t[[\hbar]]$ be an invariant star product on the nondegenerate triangular Lie bialgebra $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ over $\mathbb{K}_t$. Let $A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}$ be the *triangular Hopf QUE algebra* obtained by a twist via $F^{-1}(t)$ from the trivial triangular Hopf QUE algebra $\big(\mathcal{U}\mathfrak{a}_t[[\hbar]], \Delta_{\mathfrak{a}_t}, R_{\mathfrak{a}_t} = 1 \otimes 1\big)$. Then, this algebra is *a quantization of the pair* $(\mathfrak{a}_t, r_t)$.

The following proposition does not depend on any specific context of quantization but only on: (i) the notion of deformation of associative algebras; (ii) the fact that the Hochschild cohomology of the bialgebra $\mathcal{U}\mathfrak{a}_t$ is $H^k(\mathcal{U}\mathfrak{a}_t) = \Lambda^k\mathfrak{a}_t, k \in \mathbb{N}$, see [2]; (iii) the Hochschild cohomological interpretation of quantum Yang Baxter equation [14, 15].

Proposition 4.1. *[14] Let $F(t) = \sum_i^\infty F_i(t)\hbar^i$ and $F'(t) = \sum_i^\infty F_i'(t)\hbar^i$ be invariant star products on $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$. Let $A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}$ and $A_{\mathfrak{a}_t[[\hbar]],F'^{-1}(t)}$ be as above in this section. Suppose that $F(t)$ and $F'(t)$ coincide up to order k, i.e. $F_l(t) = F_l'(t), l = 1, 2, \ldots, k$. Then (a) there exist $h_{k+1} \in \mathfrak{a}_t \wedge \mathfrak{a}_t$ and $E_{k+1}(t) \in \mathcal{U}\mathfrak{a}_t$ such that $F_{k+1}'(t) - F_{k+1}(t) = h_{k+1} + d_H E_{k+1}(t)$ where d_H is the coboundary operator in the Hochschild cohomology of $\mathcal{U}\mathfrak{a}_t$; (b) h_{k+1} is not only a Hochschild 2-cocycle but also a Poisson 2-cocycle relatively to the invariant Poisson structure defined by the element $r_t \in \mathfrak{a}_t \wedge \mathfrak{a}_t$.*

Again, the above Hochschild cohomology spaces and proposition 4.1 play a central role in the proof of the next theorem. In the context of quantification in [8] this theorem corresponds to a main theorem by Drinfeld in the context of quantification in [5]. In [14, 15] there is a proof of this Drinfeld theorem. See the references in [15] for a similar theorem about star products on general symplectic manifolds and on Poisson manifolds.

Theorem 4.2 [16, 18]**.** *Fix a Lie associator* Φ. *Let* $A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}$ *be defined at the beginning of this section. We have (a) there exist elements* $r_{t,\hbar} = r_t + r_{t,2}\hbar + r_{t,3}\hbar^2 + \ldots \in (\wedge^2\mathfrak{a}_t)[[\hbar]]$ *and* $E^{r_{t,\hbar}} = 1 + E_1^{r_{t,\hbar}}\hbar + \cdots + E_n^{r_{t,\hbar}}\hbar^n + \cdots \in \mathcal{U}\mathfrak{a}_t[[\hbar]]$ *such that*

$$F(t) = \Delta_{\mathfrak{a}_t}((E^{r_{t,\hbar}})^{-1}) \cdot_t \tilde{J}^{\Phi}_{r_t(\hbar)} \cdot_t (E^{r_{t,\hbar}} \otimes E^{r_{t,\hbar}});$$

i.e., $F(t)$ *and* $\tilde{J}^{\Phi}_{r_t,\hbar} \in \big(\mathcal{U}\mathfrak{a}_t \otimes \mathcal{U}\mathfrak{a}_t\big)[[\hbar]]$ *are equivalent invariant star products over the nondegenerate triangular Lie bialgebra* $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ *over the ring* $\mathbb{K}_t$. *(b) The triangular Hopf QUE algebras* $A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}$ *and* $A_{\mathfrak{a}_t[[\hbar]],(\tilde{J}^{\Phi}_{r_{t,\hbar}})^{-1}}$ *are isomorphic.*

As a consequence we have the following isomorphisms:

Corollary 4.3. *Let* Φ, Φ' *be two Lie associators. Let* $A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}$ *be given as in the theorem. Let* $r_{t,\hbar}, r'_{t,\hbar} \in (\wedge^2\mathfrak{a}_t)[[\hbar]]$ *be the elements determined in the theorem by the pairs* $\big(\Phi; A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}\big)$ *and* $\big(\Phi'; A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)}\big)$, *respectively. Then we have*

$$A_{\mathfrak{a}_t[[\hbar]],F^{-1}(t)} \overset{\text{isom}}{\approx} A_{\mathfrak{a}_t[[\hbar]],(\tilde{J}^{\Phi}_{r_{t,\hbar}})^{-1}} \overset{\text{isom}}{\approx} A_{\mathfrak{a}_t[[\hbar]],(\tilde{J}^{\Phi'}_{r'_{t,\hbar}})^{-1}}.$$

5. Some properties of nondegenerate triangular Lie bialgebras $\big(\mathfrak{a}_{t,u},[;]_{\mathfrak{a}_{t,u}},\varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\big)$ over $\mathbb{K}_t[[u]]$

(1) We now develop what we wrote in section 1, (8) in the introduction. We need the following:

Proposition 5.1. *Let* $\big(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\big)$ *be a nondegenerate triangular Lie bialgebra over* $\mathbb{K}_{t,u}$. *Let* $\varphi^1_{t,u} : \mathfrak{a}_{t,u} \longrightarrow \mathfrak{a}_{t,u}$ *be a Lie algebra isomorphism. Let* $r'_{t,u}$ *be the element in* $\mathfrak{a}_t[[u]] \wedge \mathfrak{a}_t[[u]]$ *defined as* $r'_{t,u} = \big(\varphi^1_{t,u} \otimes \varphi^1_{t,u}\big)r_{t,u}$.

(a) The set $\big(\mathfrak{a}_{t,u}, [,]_{\mathfrak{a}_{t,u}}, \epsilon'_{\mathfrak{a}_{t,u}} = d_c(t)r'_{t,u}\big)$ *is a nondegenerate triangular Lie bialgebra.*

(b) The transposed map $\big(\varphi^1_{t,u}\big)^{\mathrm{t}} : \mathfrak{a}^*_{r'_{t,u}} \longrightarrow \mathfrak{a}^*_{r_{t,u}}$ *is a Lie algebra isomorphism.*

(c) The pair $\big(\varphi^1_{t,u}; \varphi^2_{t,u} = \big(\big(\varphi^1_{t,u}\big)^{\mathrm{t}}\big)^{-1}\big)$ *defines a Lie bialgebra isomorphism between the Lie bialgebra* $\big(\mathfrak{a}_t \oplus \mathfrak{a}^*_{r_{t,u}}, [,]_{\mathfrak{a}_t\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_t\oplus\mathfrak{a}^*_{r_{t,u}}} = d_c(t,u)r\big)$ *and the Lie bialgebra* $\big(\mathfrak{a}_t \oplus \mathfrak{a}^*_{r'_{t,u}}, [,]_{\mathfrak{a}_t\oplus\mathfrak{a}^*_{r'_{t,u}}}, \varepsilon_{\mathfrak{a}_t\oplus\mathfrak{a}^*_{r'_{t,u}}} = d_c(t,u)r\big)$. *Furthermore, this isomorphism sends the canonical element* $r \in (\mathfrak{a} \oplus \mathfrak{a}^*)^{\otimes 2}$ *into itself.*

Corollary 5.2.

(a) Under the hypotheses of the proposition let $\beta_{t,u} = \mu_{r_{t,u}}(r_{t,u})$ *and* $\beta'_{t,u} = \mu_{r'_{t,u}}(r'_{t,u})$ *be elements of* $\mathfrak{a}^*_t \wedge \mathfrak{a}^*_t[[u]]$. *Then* $\big(\varphi^2_{t,u} \otimes \varphi^2_{t,u}\big)\beta_{t,u} = \beta'_{t,u}$.

(b) Conversely, let $\beta_{t,u}$ *and* $\beta'_{t,u}$ *be as considered in (a). Let* $\varphi^1_{t,u} : \mathfrak{a}_{t,u} \longrightarrow \mathfrak{a}_{t,u}$ *be a Lie algebra isomorphism and* $\varphi^2_{t,u} = \big(\big(\varphi^1_{t,u}\big)^{\mathrm{t}}\big)^{-1}$. *Suppose that* $\big(\varphi^2_{t,u} \otimes \varphi^2_{t,u}\big)\beta_{t,u} = \beta'_{t,u}$. *Then,* $\big(\varphi^1_{t,u} \otimes \varphi^1_{t,u}\big)r_{t,u} = r'_{t,u}$.

(2) In the Lie algebra $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}\right)$ over $\mathbb{K}_{t,u}$ consider the following Lie-algebra isomorphisms: $\varphi^1_{t,u} = \exp\left(t\cdot ad_{X_{t,u}}\right)$ where $X_{t,u} = X_{1,u} + X_{2,u}t + X_{3,u}t^2 + \cdots \in \mathfrak{a}_u[[t]]$. Then $\varphi^2_{t,u} = \exp\left(-t\cdot ad^{\mathrm{t}}_{X_{t,u}}\right) = \exp\left(t\cdot ad^*_{X_{t,u}}\right)$. Our interest is in the map $\varphi^2_{t,u}\otimes\varphi^2_{t,u} = \exp\left(ad^*_{tX_{t,u}}\otimes 1 + 1\otimes ad^*_{tX_{t,u}}\right)$.

Proposition 5.3. *Let $\beta_{t,1} = \beta_t$ and let $\beta_{t,u} = \beta_{t,1} + \beta_{t,2}u + \beta_{t,3}u^2 + \ldots \in \wedge^2(\mathfrak{a}^*_t[[u]])$, or equivalently $\beta_{t,u} = \beta_{1,u} + \beta_{2,u}t + \beta_{3,u}t^2 + \ldots \in \wedge^2(\mathfrak{a}^*_u[[t]])$, be a nondegenerate 2-cocycle on the Lie algebra $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}\right)$. The elements $\beta_{t,1}, \beta_{t,2}, \ldots \in \wedge^2(\mathfrak{a}^*_t)$ are then 2-cocycles on the Lie algebra $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right)$, with the trivial action, and $\beta_{t,1}$ is nondegenerate. Let $X_{t,u}$ be as considered before. Then*

$$\exp\left(ad^*_{tX_{t,u}}\right)^{\otimes^2}(\beta_{t,u}) = \exp\left(ad^*_{tX_{t,u}}\otimes 1 + 1\otimes ad^*_{tX_{t,u}}\right)(\beta_{t,u}) = \beta_{t,u} + \mathrm{d}_R(t)\gamma_{t,u},$$

where $\gamma_{t,u} = t\sum_{l\geqslant 1}\gamma_{l,u}(t)t^{l-1} \in t\mathfrak{a}^*_u[[t]]$ and

$$\gamma_{1,u}(t) = (i(X_{t,u})\beta_{t,u})$$
$$\gamma_{2,u}(t) = \left(\frac{1}{2!}(i(X_{t,u})\beta_{t,u})\circ ad_{X_{t,u}}\right)$$
$$\gamma_{3,u}(t) = \left((i(X_{t,u})\beta_{t,u})\circ ad_{X_{t,u}}\circ ad_{X_{t,u}}\right), etc.$$

A converse of proposition 5.3 is

Proposition 5.4. *Let $\beta_{t,u}$ be as considered in proposition 5.3. Let $\gamma_{t,u} = \alpha_{1,u}t + \alpha_{2,u}t^2 + \alpha_{3,u}t^3 + \ldots \in \mathfrak{a}^*_u[[t]]$; $\alpha_{l,u}\in\mathfrak{a}^*_u$, $l = 1, 2, \ldots$. Define $\beta'_{t,u} = \beta_{t,u} + d_R(t)\gamma_{t,u}$. Then, there exists a unique $X_{t,u} = X_{1,u} + X_{2,u}t + X_{3,u}t^2 + \ldots \in \mathfrak{a}_u[[t]]$ such that $\exp\left(ad^*_{tX_{t,u}}\right)^{\otimes^2}(\beta_{t,u}) = \beta'_{t,u}$. It is given by*

$$i(X_{1,u})\beta_{1,u} = \alpha_{1,u},$$
$$i(X_{2,u})\beta_{1,u} + i(X_{1,u})\beta_{2,u} + \frac{1}{2!}(i(X_1)\beta_1)\circ(i(X_{1,u})B_{1,u}) = \alpha_{2,u},$$
$$\begin{aligned} &i(X_{1,u})\beta_{3,u} + i(X_{2,u})\beta_{2,u} + i(X_{3,u})\beta_{1,u}\\ &\quad + \frac{1}{2!}\big((i(X_{2,u})\beta_{1,u})\circ(i(X_{1,u})B_{1,u}) + (i(X_{1,u})\beta_{2,u})\circ(i(X_{1,u})B_{1,u})\\ &\quad + (i(X_{1,u})\beta_{1,u})\circ(i(X_{2,u})B_{1,u}) + (i(X_{1,u})\beta_{1,u})\circ(i(X_{1,u})B_{2,u})\\ &\quad + (i(X_{1,u})\beta_{1,u})\circ(i(X_{1,u})B_{1,u})\circ(i(X_{1,u})B_{1,u})\big) = \alpha_{3,u}, \quad etc. \end{aligned}$$

where $B_{l,u}\in L(\mathfrak{a}_u, \mathfrak{a}_u; \mathfrak{a}_u)$, $l = 1, 2, \ldots$ are some well-determined bilinear mappings.

As $\beta_{1,u}$ is invertible the first equation allows us to compute $X_{1,u}$. Analogously the second equation allows us to obtain $X_{2,u}$ etc. It is easy to obtain a general form for $X_{l,u}$ as a function of $X_{k,u}$, $1\leqslant k < l$.

(3) The following property is needed:

Proposition 5.5. *Let $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t)r_{t,u}\right)$ and $\left(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon'_{\mathfrak{a}_{t,u}} = d_c(t)r'_{t,u}\right)$ be nondegenerate triangular Lie bialgebras as considered in section 2. Let $\left(\mathfrak{a}_{t,u}\oplus\mathfrak{a}_{r_{t,u}}, [;]_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}} = d_c(t,u)r\right)$ and $\left(\mathfrak{a}_{t,u}\oplus\mathfrak{a}_{r'_{t,u}}, [;]_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}}, \varepsilon_{\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r'_{t,u}}} = d_c(t,u)r\right)$ be the corresponding quasitriangular doubles. Let $\left(\varphi^1_{t,u};\psi_{t,u}\right):\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}\longrightarrow\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r'_{t,u}}$ be a Lie algebra isomorphism such that $\varphi^1_{t,u}:\mathfrak{a}_{t,u}\longrightarrow\mathfrak{a}_{t,u}$ and $\psi_{t,u}:\mathfrak{a}^*_{r_{t,u}}\longrightarrow\mathfrak{a}^*_{r'_{t,u}}$ are Lie algebra isomorphisms. Let $\tilde\varphi^1_{t,u}, \tilde\psi_{t,u}$ be the extensions of $\varphi^1_{t,u}$ and $\psi_{t,u}$ to homomorphisms $\mathcal{U}\mathfrak{a}_{t,u}\longrightarrow\mathcal{U}\mathfrak{a}_{t,u}$ and $\mathcal{U}\mathfrak{a}^*_{r_{t,u}}\longrightarrow\mathcal{U}\mathfrak{a}^*_{r'_{t,u}}$. Let $X(t,u)\in\mathcal{U}\left(\mathfrak{a}_{t,u}\oplus\mathfrak{a}^*_{r_{t,u}}\right)^{\otimes^2}$. Let $\phi_{r_{t,u}}$ and $\phi_{r'_{t,u}}$ be the Lie algebra-module isomorphisms defined in theorem 2.3, (2). Then we have*

$$\phi^{-1}_{r'_{t,u}}\left[\left(\left(\tilde\varphi^1_{t,u};\tilde\psi_{t,u}\right)^{\otimes^2}(X(t,u))\right)\cdot\left(1_+^{r'_{t,u}}\otimes 1_-^{r'_{t,u}}\right)\right] = \left(\left(\tilde\varphi^1_{t,u};\tilde\psi_{t,u}\right)\circ\phi^{-1}_{r_{t,u}}\right)\left(X(t,u)\cdot\left(1_+^{r_{t,u}}\otimes 1_-^{r_{t,u}}\right)\right).$$

We can also prove the following:

Proposition 5.6. *Let* $(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}}, \varepsilon_{\mathfrak{a}_{t,u}} = d_c(t,u)r_{t,u})$ *and* $(\mathfrak{a}_{t,u}, [,]_{\mathfrak{a}_{t,u}}, \varepsilon'_{\mathfrak{a}_{t,u}} = d_c(t,u)r'_{t,u})$ *be nondegenerate triangular Lie bialgebras over* $\mathbb{K}_{t,u}$. *Let* $\varphi^1_{t,u} : \mathfrak{a}_{t,u} \longrightarrow \mathfrak{a}_{t,u}$ *be a Lie algebra isomorphism such that* $r'_{t,u} = (\varphi^1_{t,u} \otimes \varphi^1_{t,u})r_{t,u}$ *and let* $(\varphi^1_{t,u}; \varphi^2_{t,u})$ *be the Lie bialgebra isomorphism between the corresponding classical doubles constructed in proposition 5.1. Then, we have*

$$\widetilde{\pi}'_{t,u} \circ (\varphi^1_{t,u}; \varphi^2_{t,u}) = \varphi^1_{t,u} \circ \tilde{\pi}_{t,u},$$

where $\tilde{\pi}'_{t,u}$ *and* $\tilde{\pi}_{t,u}$ *are defined in proposition 3.1.*

6. A necessary and sufficient condition to be isomorphic two triangular Hopf algebras $A_{\mathfrak{a}_t[[\hbar]],\tilde{J}^{-1}_{r_{t,\hbar}}}$ and $A_{\mathfrak{a}_t[[\hbar]],\tilde{J}^{-1}_{r'_{t,\hbar}}}$ over $\mathbb{K}_t[[\hbar]]$

If in the expression of $J_{r_{t,u}}$ given in theorem 2.3 we take into account proposition 5.5 and also the form of a Lie associator $\Phi = \mathrm{e}^{P(\hbar\Omega_{12},\hbar\Omega_{23})}$ we arrive at

Proposition 6.1. *Hypotheses are as in proposition 5.5. Let* $J_{r'_{t,u}}$ *and* $J_{r_{t,u}}$ *be the corresponding elements in theorem 2.3, (3). Suppose moreover that* $(\varphi^1_{t,u}; \psi_{t,u}) \otimes (\varphi^1_{t,u}; \psi_{t,u})\Omega = \Omega$. *Then* $J_{r'_{t,u}} = (\tilde{\varphi}^1_{t,u}; \tilde{\psi}_{t,u})^{\otimes^2} J_{r_{t,u}}$. *In particular, this proposition is valid for the Lie bialgebra isomorphism* $(\varphi^1_{t,u}; \varphi^2_{t,u})$ *considered in propositions 5.3 and 5.4.*

(4) Using propositions 5.1, 6.1, 5.6 and corollary 5.2 we can prove

Proposition 6.2. *(a) Let* $\tilde{J}_{r_{t,\hbar}}$ *and* $\tilde{J}_{r'_{t,\hbar}}$ *be elements* $\in (\mathcal{U}\mathfrak{a}_t)^{\otimes 2}[[\hbar]]$ *which are invariant star products on a nondegenerate triangular Lie bialgebra* $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ *over* $\mathbb{K}_t$ *as in definition 3.5, and obtained as in corollary 3.3 and corollary 3.9, respectively from the nondegenerate solutions* $r_{t,u}, r'_{t,u} \in (\mathfrak{a}_t \wedge \mathfrak{a}_t)[[u]]$ *of the YBE on the Lie-algebra* $(\mathfrak{a}_{t,u}, [;]_{\mathfrak{a}_{t,u}})$ *as in section 1. Let* $\mu_{r_{t,u}}(r_{t,u}) = \beta_{t,u} = \beta_{t,1} + \beta_{t,2}u + \beta_{t,3}u^2 \ldots \in (\mathfrak{a}^*_t \wedge \mathfrak{a}^*_t)[[u]]$ *and* $\mu_{r'_{t,u}}(r'_{t,u}) = \beta'_{t,u} = \beta_{t,1} + \beta'_{t,2}u + \beta'_{t,3}u^2 \ldots \in (\mathfrak{a}^*_t \wedge \mathfrak{a}^*_t)[[u]]$. *(b) Suppose that the cocycles* $\beta_{t,u}$ *and* $\beta'_{t,u}$ *belong to the same cohomological class in* $H(\mathfrak{a}_{t,u}) \equiv H^2(\mathfrak{a}_t)[[u]]$, *i.e.,* $\beta'_{t,u} = \beta_{t,u} + d_R(t)\gamma_{t,u}$ *for some 1-cochain* $\gamma_{t,u} = \gamma_{t,1}u + \gamma_{t,2}u^2 + \gamma_{t,3}u^3 \cdots \in \mathfrak{a}^*_t[[u]]$. *Then,* $\tilde{J}_{r_{t,\hbar}}$ *and* $\tilde{J}_{r'_{t,\hbar}}$ *are equivalent invariant star products.*

To prove the converse we need the following lemma:

Lemma 6.3. *Suppose that in proposition 6.2*

$$\beta_{t,u} = \beta_{t,1} + \beta_{t,2}u + \beta_{t,3}u^2 + \cdots + \beta_{t,R-1}u^{R-2} + \beta_{t,R}u^{R-1} + \cdots$$
$$\beta'_{t,u} = \beta_{t,1} + \beta_{t,2}u + \beta_{t,3}u^2 + \cdots + \beta_{t,R-1}u^{R-2} + (\beta_{t,R} + d_R\alpha_{t,(R-1)})u^{R-1} + \cdots,$$

where $\alpha_{t,(R-1)}$ *is an element in* $\mathfrak{a}^*_t$, *that is, a 1-cochain on the Lie algebra* $(\mathfrak{a}_t, [;]_{\mathfrak{a}_t})$ *over the ring* $\mathbb{K}_t$. *This means that* $\beta'_{t,u}$ *and* $\beta_{t,u}$ *are equal except in the term of order* $R-1$. *Then,* $\tilde{J}_{r_t,\hbar}$ *and* $\tilde{J}_{r'_t,\hbar}$ *are equivalent,*

$$\tilde{J}_{r'_{t,\hbar}} = \Delta_{\mathfrak{a}_t}(E)^{-1} \cdot_t \tilde{J}_{r_{t,\hbar}} \cdot_t (E \otimes E),$$

and the element $E = 1 + E_{t,1}\hbar + E_{t,2}\hbar^2 + \cdots + E_{t,(R-1)}\hbar^{R-1} + \cdots \in \mathcal{U}\mathfrak{a}_t[[\hbar]]$ *which defines this equivalence verifies*

$$E_{t,1} = 0, \quad E_{t,2}, \ \ldots, \quad E_{t,(R-2)} = 0, \quad E_{t,(R-1)} = \chi_{r_t}(\alpha_{t,(R-1)}) = \mu^{-1}_{r_t}(\alpha_{t,(R-1)}).$$

Lemma 6.3 and Hochschild cohomology properties allow us to prove

Proposition 6.4. *Let $\tilde{J}_{r_{t,\hbar}}$ and $\tilde{J}_{r'_{t,\hbar}}$ be as considered in theorem 6.2. Suppose that $\tilde{J}_{r_{t,\hbar}}$ and $\tilde{J}_{r'_{t,\hbar}}$ are equivalent. Then $\beta'_{t,u}$ and $\beta_{t,u}$ belong to the same cohomological class, i.e., $\beta'_{t,u} = \beta_{t,u} + d_R(t)\gamma_{t,u}$ for some 1-cochain $\gamma_{t,u} \cdots \in \mathfrak{a}_t^*[[u]]$.*

Combining the last two theorems we obtain the following result, *similar in Etingof–Kazhdan quantization theory to that by Drinfeld in* [5]:

Proposition 6.5. *Let $\tilde{J}_{r_{t,\hbar}}$ and $\tilde{J}_{r'_{t,\hbar}}$ be elements in $(\mathcal{U}\mathfrak{a}_t)^{\otimes 2}[[\hbar]]$ which are invariant star products on a nondegenerate triangular Lie bialgebra $(\mathfrak{a}_t, [,]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t)r_t)$ over $\mathbb{K}_t$ as in theorem 6.2. Then $\tilde{J}_{r_{t,\hbar}}$ and $\tilde{J}_{r'_{t,\hbar}}$ are equivalent invariant star products if, and only if, $\mu_{r_{t,u}}(r_{t,u}) = \beta_{t,u}$ and $\mu_{r'_{t,u}}(r'_{t,u}) = \beta'_{t,u}$ belong to the same cohomological class in $H^2(\mathfrak{a}_t)[[u]]$. In other words, $\tilde{J}_{r_{t,\hbar}}$ and $\tilde{J}_{r'_{t,\hbar}}$ are equivalent invariant star products if, and only if, there exists a 1-cochain $\gamma_{t,u} \in \mathfrak{a}_{t,u}^*$ such that $\beta'_{t,u} = \beta_{t,u} + d_R(t)\gamma_{t,u}$.*

Proposition 6.5 and remark (2) on page 841 of [7] allow us to obtain

Proposition 6.6. *Two triangular Hopf algebras $A_{\mathfrak{a}_t[[\hbar]],\tilde{J}^{-1}_{r_{t,\hbar}}}$ and $A_{\mathfrak{a}_t[[\hbar]],\tilde{J}^{-1}_{r'_{t,\hbar}}}$ over $\mathbb{K}_t[[\hbar]]$ defined as in corollary 3.9 and section 4 are isomorphic if, and only if, $\mu_{r_{t,u}}(r_{t,u}) = \beta_{t,u}$ and $\mu_{r'_{t,u}}(r'_{t,u}) = \beta'_{t,u}$ belong to the same cohomological class in $H^2(\mathfrak{a}_t)[[u]]$.*

(5) From the above results and remark (2) on page 841 of [7] we may also prove

Proposition 6.7. *Let $A_{\mathfrak{a}_{t,\hbar}\oplus\mathfrak{a}^*_{r_{t,\hbar}},\Omega,J^{-1}_{r_{t,\hbar}}}$ and $A_{\mathfrak{a}_{t,\hbar}\oplus\mathfrak{a}^*_{r'_{t,\hbar}},\Omega,J^{-1}_{r'_{t,\hbar}}}$ be quasitriangular Hopf QUE algebras over $\mathbb{K}_t[[\hbar]]$ which are quantizations, as in theorem 2.3 after putting $u = \hbar$, of the quasitriangular Lie bialgebra $\left(\mathfrak{a}_t \oplus \mathfrak{a}^*_{r_t}, [,]_{\mathfrak{a}_t\oplus\mathfrak{a}^*_{r_t}}, \varepsilon_{\mathfrak{a}_t\oplus\mathfrak{a}^*_{r_t}} = d_c(t)r\right)$ over the ring $\mathbb{K}_t$. Suppose that the cocycles $\mu_{r_{t,u}}(r_{t,u}) = \beta_{t,u}$ and $\mu_{r'_{t,u}}(r'_{t,u})$ as in proposition 6.2 define the same cohomological class in $H^2(\mathfrak{a}_t)[[u]]$. Then the quasitriangular Hopf QUE algebras $A_{\mathfrak{a}_{t,\hbar}\oplus\mathfrak{a}^*_{r_{t,\hbar}},\Omega,J^{-1}_{r_{t,\hbar}}}$ and $A_{\mathfrak{a}_{t,\hbar}\oplus\mathfrak{a}^*_{r'_{t,\hbar}},\Omega,J^{-1}_{r'_{t,\hbar}}}$ over the ring $\mathbb{K}_t[[\hbar]]$ are isomorphic.*

7. Isomorphic triangular Hopf algebras over $\mathbb{K}[[\hbar]]$ of type $A_{\mathfrak{a}_\hbar,\tilde{J}_{r_{\hbar,\hbar}}}$

We start from a *deformation algebra* $(\mathfrak{a}_\hbar, [;]_{\mathfrak{a}_\hbar})$ of the Lie algebra $(\mathfrak{a}, [;]_{\mathfrak{a}})$ over the field $\mathbb{K}$ and from *an element* $r_\hbar = \sum_{l\geqslant 1} r_l\hbar^{l-1} \in \mathfrak{a}_\hbar \wedge \mathfrak{a}_\hbar$ *which is nondegenerate* (r_1 is invertible) and a solution of the YBE ($[r_\hbar, r_\hbar]_{\mathfrak{a}_\hbar} = 0$) on the Lie algebra $(\mathfrak{a}_\hbar, [;]_{\mathfrak{a}_\hbar})$ over $\mathbb{K}[[\hbar]]$. We call the set $(\mathfrak{a}_\hbar, [;]_{\mathfrak{a}_\hbar}, r_\hbar)$ *a nondegenerate triangular Lie bialgebra deformation of the nondegenerate triangular Lie bialgebra* $(\mathfrak{a}, [,]_{\mathfrak{a}}, r_1)$ *over* $\mathbb{K}$. These elements are just the elements $\left(\mathfrak{a}_t, [;]_{\mathfrak{a}_t}\right), r_t = \sum_{l\geqslant 1} r_l t^{l-1} \in \mathfrak{a}_t \wedge \mathfrak{a}_t, [r_t, r_t]_{\mathfrak{a}_t} = 0$ considered before setting $t = \hbar$. Consider two elements $r_{t,u}$ and $r'_{t,u}$ as in section 6. We set $u = \hbar$ and obtain $r_{t,\hbar}$ and $r'_{t,\hbar}$, as in that section, and the corresponding propositions there. We set moreover $t = \hbar$ and get $r_{\hbar,\hbar}, r'_{\hbar,\hbar} \in \mathfrak{a}_\hbar \wedge \mathfrak{a}_\hbar$. *The corresponding elements* $\tilde{J}^{-1}_{r_{\hbar,\hbar}}, \tilde{J}^{-1}_{r_{\hbar,\hbar}} \in \mathcal{U}\mathfrak{a}_\hbar \hat{\otimes} \mathcal{U}\mathfrak{a}_\hbar$ will also be called invariant star products on the deformation algebra $(\mathfrak{a}_\hbar, [;]_{\mathfrak{a}_\hbar})$. From proposition 6.5 we obtain

Proposition 7.1. *Let $\tilde{J}_{r_{\hbar,\hbar}}$ and $\tilde{J}_{r'_{\hbar,\hbar}} \in \mathcal{U}\mathfrak{a}_\hbar \hat{\otimes} \mathcal{U}\mathfrak{a}_\hbar$ be the above invariant star products on a non-degenerate triangular Lie bialgebra deformation $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar)r_\hbar)$ of the non-degenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ over $\mathbb{K}$. Let $\mu_{r_{t,\hbar}}(r_{t,\hbar}) = \beta_{t,\hbar}$ and $\mu_{r'_{t,\hbar}}(r'_{t,\hbar}) = \beta'_{t,\hbar}$ be the elements defining them and $A_{\mathfrak{a}_\hbar,\tilde{J}_{r_{\hbar,\hbar}}}$ and $A_{\mathfrak{a}_\hbar,\tilde{J}_{r'_{\hbar,\hbar}}}$ the corresponding triangular quantized universal enveloping algebras over $\mathbb{K}[[\hbar]]$. These algebras* *are isomorphic if, and only if, $\beta_{t,\hbar}$ and $\beta'_{t,\hbar}$ belong to the same cohomological class in $H^2_R(\mathfrak{a}_t)[[\hbar]]$.*

Also, we obtain

Corollary 7.2. *Suppose that the deformation algebra $(\mathfrak{a}_\hbar, [;]_{\mathfrak{a}_\hbar})$ is a trivial one, that is, it is just that obtained by extension of scalars $\mathbb{K} \longrightarrow \mathbb{K}[[\hbar]]$. Suppose also that $r_\hbar = r_1$ and $r_{\hbar,\hbar} = r_1 + r_2\hbar + r_3\hbar^2 + \cdots$ and write it as $r_{0,\hbar}$. Similarly suppose $r'_\hbar = r_1$ and $r'_{\hbar,\hbar} = r_1 + r'_2\hbar + r'_3\hbar^2 + \cdots$ and write it as $r'_{0,\hbar}$. Let $\beta_{0,\hbar} = \beta_1 + \beta_2\hbar + \beta_3\hbar^2 + \cdots$ and $\beta'_{0,\hbar} = \beta_1 + \beta'_2\hbar + \beta'_3\hbar^2 + \cdots$ be the corresponding elements in $(\mathfrak{a}^* \wedge \mathfrak{a}^*)[[\hbar]]$. We have*

(i) If β_k and β'_k are, for any $k = 2, 3, 4, \ldots$, in the same cohomological class in $H^2(\mathfrak{a})$, the triangular Hopf quantized universal enveloping algebras $A_{\mathfrak{a}_\hbar, \tilde{J}_{r_{0,\hbar}}}$ and $A_{\mathfrak{a}_\hbar, \tilde{J}_{r'_{0,\hbar}}}$ are isomorphic and conversely.

(ii) In the particular case when $\mathbb{K}$ is the field $\mathbb{R}$, what we get is that the set of equivalent classes of quantizations (usual term) of the Lie group $\mathbb{G}$ with Lie algebra $(\mathfrak{a}, [,]_{\mathfrak{a}})$ and endowed with a left invariant symplectic structure $\beta_1 \in \mathfrak{a}^* \wedge \mathfrak{a}^*$ is in a bijective correspondence with the set $\beta_1 + \hbar H^2(\mathfrak{a})[[\hbar]]$. A theorem in the Etingof–Kazhdan setting similar to that given by Drinfeld [5].

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